Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 343 (2008) 99–109

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Quasi-parabolic analytic transformations of $\mathbf{C}^n$

Feng Rong

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

Received 10 August 2007

Available online 15 January 2008

Submitted by J. Noguchi

Abstract

Let F be an analytic transformation of $\mathbf{C}^n$, with the origin O as a quasi-parabolic fixed point. We will associate an invariant, order $\nu(F)$, to F and study the local dynamics of F when it has a non-degenerate characteristic direction $[v]$ and is dynamically separating in the direction $[v]$. We show that for such F there exist at least $\nu(F)-1$ parabolic curves tangent to $[v]$ at the origin. We also study a non-dynamically-separating example in details.

Keywords: Quasi-parabolic analytic transformations; Parabolic curves

1. Introduction

Let F be a germ of analytic transformation of $(\mathbf{C}^n, O)$. We are interested in understanding the dynamics near O of F for which the spectrum of the differential dF_O contains 1. The most studied case is when F is tangent to the identity, that is $dF_O = id$. (See [7,8,17] and [2] for precise results and [1] and [3] for more information.) When dF_O is not the identity, but it still has 1 as eigenvalue, two cases have received some attention. The first is the semi-attractive case, where the spectrum of dF_O contains 1 and λ_j's with $|\lambda_j| < 1$. (See [5,6,10,12,15,16].) In this paper, we study the case when O is a quasi-parabolic fixed point for F, that is the spectrum of dF_O contains 1 and λ_j's with $|\lambda_j| = 1$ and $\lambda_j \neq 1$. This case has been studied by several authors [4,11,13]. In [11], Pöschel proved the existence of an F-invariant complex manifold through O tangent to the eigenspace of λ_j's on which F is holomorphically linearizable when λ_j's satisfy some Brjuno condition. Using similar ideas, the author [13] showed the existence of "Siegel cylinders" for certain quasi-parabolic germs when λ_j's satisfy some Brjuno condition. In [4], Bracci and Molino studied quasi-parabolic germs in $\mathbf{C}^2$ and showed the existence of "parabolic curves" at O tangent to the eigenspace of 1 for germs they called "dynamically separating." We are going to study quasi-parabolic germs in $\mathbf{C}^n$ ($n > 1$), using ideas of Hakim [7] and Bracci and Molino [4]. Before we state our results, let us recall some definitions and known results.

E-mail address: frong@umich.edu.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.01.014

Definition 1.1. A *parabolic curve* for F, a germ of analytic transformation of $(\mathbf{C}^n, O)$, is an injective holomorphic map $\varphi : \Delta \to \mathbf{C}^n$ satisfying the following properties:

(i) Δ is a simply connected domain in $\mathbf{C}$ with $0 \in \partial\Delta$;
(ii) φ is continuous on $\partial\Delta$ and $\varphi(0) = O$;
(iii) $\varphi(\Delta)$ is invariant under F and $F^k(\varphi(\zeta)) \to O$ as $k \to \infty$ for any $\zeta \in \Delta$.

Furthermore, if $[\varphi(\zeta)] \to [v] \in \mathbf{P}^{n-1}$ as $\zeta \to 0$ (where $[\cdot]$ denotes the canonical projection of $\mathbf{C}^n \backslash \{O\}$ onto $\mathbf{P}^{n-1}$), we say that φ is *tangent to* $[v]$ at the origin.

Definition 1.2. Let F be a germ of analytic transformation of $(\mathbf{C}^n, O)$ tangent to the identity. Let $F = id + F_2 + F_3 + \cdots$ be the homogeneous expansion of F in series of homogeneous polynomials. Then the *order* of F is $\nu(F) = \min\{j \mid F_j \not\equiv 0\}$ and a *characteristic direction* for F is a vector $[v] \in \mathbf{P}^{n-1}$ such that there is $\lambda \in \mathbf{C}$ so that $F_{\nu(F)}(v) = \lambda v$. If $\lambda \neq 0$ we say that $[v]$ is *non-degenerate*, otherwise it is *degenerate*.

In [7], Hakim proved the following theorem.

Theorem 1.3. *Let F be a germ of analytic transformation of $(\mathbf{C}^n, O)$ tangent to the identity of order $\nu(F)$. Then for every non-degenerate characteristic direction $[v]$ of F there exist at least $\nu(F) - 1$ parabolic curves tangent to $[v]$ at the origin.*

In [4], Bracci and Molino generalized the notion of *order* to the quasi-parabolic case in $\mathbf{C}^2$ and proved the following theorem. (See [4] for precise definitions.)

Theorem 1.4. *Let F be a germ of analytic transformation of $(\mathbf{C}^2, O)$, with the origin as a quasi-parabolic fixed point. If F is of order $\nu(F)$ and is dynamically separating then there exist at least $\nu(F) - 1$ parabolic curves tangent to the eigenspace of 1 at the origin.*

In this paper, we will generalize the notion of *order*, *characteristic direction* and *dynamically separating* to the quasi-parabolic case in any dimension, and extend the results of Hakim [7] and Bracci and Molino [4].

Our main result is the following

Theorem 1.5. *Let F be a germ of analytic transformation of $(\mathbf{C}^n, O)$, with the origin as a quasi-parabolic fixed point. If F is of order $\nu(F)$ and is dynamically separating in a non-degenerate characteristic direction $[v]$, then there exist at least $\nu(F) - 1$ parabolic curves tangent to $[v]$ at the origin.*

In Section 2, we make several definitions necessary for our study and choose suitable coordinates to get a simpler expression for F. We prove Theorem 1.5 in Section 3. Finally, we give a detailed analysis of an interesting example in Section 4.

2. Fundamental invariants

Let F be a germ of analytic transformation of $(\mathbf{C}^n, O)$, with the origin as a quasi-parabolic fixed point. Assume that dF_O is diagonalizable and $\mathrm{Spec}(dF_O) = \{1, \ldots, 1, \lambda_1, \ldots, \lambda_m\}$, with $|\lambda_j| = 1$ and $\lambda_j \neq 1$. Let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$. Then in some system of local coordinates, we can write F as

$$\begin{cases} x_1 = x + p_2(x, y, z) + p_3(x, y, z) + \cdots, \\ y_1 = y + q_2(x, y, z) + q_3(x, y, z) + \cdots, \\ z_1 = \Lambda z + r_2(x, y, z) + r_3(x, y, z) + \cdots, \end{cases} \tag{2.1}$$

where $y = (y_1, \ldots, y_l)^t$ with $l = n - m - 1$, $z = (z_1, \ldots, z_m)^t$ and p_i, q_i and r_i are vectors of homogeneous polynomials of degree i.

Set $j = (j_1, \dots, j_l)$ with j_i being non-negative integers, and $|j| = j_1 + \dots + j_l$. Similarly, set $k = (k_1, \dots, k_m)$ with k_i being non-negative integers, and $|k| = k_1 + \dots + k_m$. Using multi-index notation, we write $y^j = y_1^{j_1} \cdot \dots \cdot y_l^{j_l}$ and $z^k = z_1^{k_1} \cdot \dots \cdot z_m^{k_m}$. First, we need the following simple lemma.

Lemma 2.1. *Let F be given by* (2.1). *Then for any given integer $\mu > 1$, there exists a local holomorphic change of coordinates such that there are no terms $x^i y^j$ with $i + |j| \leqslant \mu$ in the expression of z_1.*

Proof. Denote by $Z_{i,j}$ the first non-zero vector of coefficients of terms $x^i y^j$ with $i + |j| \leqslant \mu$ in the expression of z_1. To get rid of such terms, consider the transformation

$$\begin{cases} x = X, \\ y = Y, \\ z = Z + x^i y^j S, \end{cases} \tag{2.2}$$

with $S = -(\Lambda - I_m)^{-1} Z_{i,j}$, where I_m is the identity matrix. Proceeding this way, we are done. $\square$

Definition 2.2. Let F be as in (2.1). Let ν (respectively, μ) be the least of $i + |j| > 1$ for terms $x^i y^j$ in the expression of x_1 and y_1 (respectively, z_1). If $\nu < \infty$ and $\mu \geqslant \nu$, then we say that F is *ultra-resonant*, and that the *order* of F is $\nu(F) := \nu$. If $\mu = \infty$ (without assuming anything on ν), then we say that F is *asymptotically ultra-resonant*.

We explicitly remark that if F is not the identity when restricted to $\{z = 0\}$ then (using Lemma 2.1) we can find a local holomorphic change of coordinates putting F in ultra-resonant form.

The following lemma shows that the order $\nu(F)$ is well defined. The proof is essentially the same as for [4, Lemma 2.5], which we include for completeness.

Lemma 2.3. *Let F and G be two analytic transformations in ultra-resonant form. If F is conjugated to G, then $\nu(F) = \nu(G)$.*

Proof. Writing $w = (x, y)$ and using multi-index notation, we can express F as

$$\begin{cases} w_1 = w + A_{1\nu}(w) + B_1(w, z) + H_1(w, z), \\ z_1 = \Lambda z + A_{2\mu}(w, z) + B_2(w, z) + H_2(w, z). \end{cases} \tag{2.3}$$

Here, $A_{1\nu}(w)$ contains terms w^ν with $|\nu| = \nu(F)$, $A_{2\mu}(w, z)$ contains terms $w^\mu z^k$ with $|k| = 1$ and $1 \leqslant |\mu| = \mu(F) := \min\{|\mu|\colon\ w^\mu z^k \text{ in } z_1 \text{ with } |k| = 1\}$, $B_1(w, z)$ contains terms $w^j z^k$ with $|k| \geqslant 1$ and $2 \leqslant |j| + |k| \leqslant \nu(F)$, $B_2(w, z)$ contains terms with $w^j z^k$ with $|k| \geqslant 2$ and $|j| + |k| \leqslant \mu(F) + 1$, $H_1(w, z)$ contains terms $w^j z^k$ with $|j| + |k| > \nu(F)$ and $H_2(w, z)$ contains terms w^j with $|j| > \nu(F)$ and $w^j z^k$ with $|k| \geqslant 1$ and $|j| + |k| > \mu(F) + 1$.

Similarly, we can express G as

$$\begin{cases} \tilde{w}_1 = w + \tilde{A}_{1\tilde{\nu}}(w) + \tilde{B}_1(w, z) + \tilde{H}_1(w, z), \\ \tilde{z}_1 = \Lambda z + \tilde{A}_{2\tilde{\mu}}(w, z) + \tilde{B}_2(w, z) + \tilde{H}_2(w, z). \end{cases} \tag{2.4}$$

If Ψ is the transformation which conjugates F to G, then it is easy to see that Ψ must be of the following form

$$\begin{cases} w_1' = Cw + \psi_1(w, z), \\ z_1' = Dz + \psi_2(w, z), \end{cases} \tag{2.5}$$

where C is an invertible $(n - m, n - m)$ matrix and D is an invertible (m, m) matrix.

From $F \circ \Psi = \Psi \circ G$, we obtain

$$\psi_1(w, z) + A_{1\nu}(w_1') + B_1(w_1', z_1') + H_1(w_1', z_1') = \psi_1(\tilde{w}_1, \tilde{z}_1) + C\big[\tilde{A}_{1\tilde{\nu}}(w) + \tilde{B}_1(w, z) + \tilde{H}_1(w, z)\big], \tag{2.6}$$

and

$$\begin{aligned} &\Lambda \psi_2(w, z) + A_{2\mu}(w_1', z_1') + B_2(w_1', z_1') + H_2(w_1', z_1') \\ &\quad = \psi_2(\tilde{w}_1, \tilde{z}_1) + D\big[\tilde{A}_{2\tilde{\mu}}(w, z) + \tilde{B}_2(w, z) + \tilde{H}_2(w, z)\big]. \end{aligned} \tag{2.7}$$

Set $\nu_h = \min\{|j|\colon\ w^j \text{ in } \psi_h(w,z)\}$, for $h = 1, 2$. Since $\Lambda\psi_2(w,z) - \psi_2(\tilde{w}_1, \tilde{z}_1)$ contains terms w^j with $|j| = \nu_h$ and it is easy to check that terms w^j in other terms in (2.7) have order $|j| \geqslant \min\{\nu(F), \nu(G), \nu_2 + 1\}$, we get from (2.7) that

$$\nu_2 \geqslant \min\bigl\{\nu(F), \nu(G)\bigr\}.$$

Combining this with (2.6), we get

$$\psi_1(\tilde{w}_1, \tilde{z}_1) - \psi_1(w,z) = A_{1\nu}(w_1') - C\tilde{A}_{1\tilde{\nu}}(w) + R(w,z),$$

where terms w^j in $R(w,z)$ have orders $|j| > \min\{\nu(F), \nu(G)\}$. Writing $\psi_1(w,z) = \sum_{|j|+|k|\geqslant 2} \psi_1^{j,k} w^j z^k$, one readily checks that terms w^j in $\psi_1(\tilde{w}_1, \tilde{z}_1) - \psi_1(w,z)$ have orders $|j| > \nu(G)$. Since $A_{1\nu}(w_1')$ contains terms w^j with $|j| = \nu(F)$ and $\tilde{A}_{1\tilde{\nu}}(w)$ contains terms w^j with $|j| = \nu(G)$, we see that $\nu(F) = \nu(G)$.

Note that the above argument is still valid if $\nu(F) = \infty$ or $\nu(G) = \infty$. □

Following Hakim [7], we make the following definition.

Definition 2.4. Let F be as in (2.1) and assume that F is ultra-resonant with $\nu(F) < \infty$. A *characteristic direction* for F is a vector $[v] = [v_1 : \cdots : v_n] \in \mathbf{P}^{n-1}$, with $v_i = 0$ for $l + 1 < i \leqslant n$, such that $F_{\nu(F)}(v) = \lambda v$ for some $\lambda \in \mathbf{C}$. If $\lambda \neq 0$, we say that $[v]$ is *non-degenerate*, otherwise it is *degenerate*.

Using this definition, we have the following proposition, whose proof is similar to that of [4, Proposition 1.3].

Proposition 2.5. *Let F be as in* (2.1)*. Then there exists an invariant nonsingular complex manifold M of dimension $n - m$ for F passing through O and tangent to the eigenspace of* 1 *if and only if F is analytically conjugated to an asymptotic ultra-resonant form. Moreover in this case, if $\nu(F) = \infty$ then F pointwise fixes M, while if $\nu(F) < \infty$ and* $[v]$ *is a non-degenerate characteristic direction for F then there exist at least $\nu(F) - 1$ parabolic curves for F at O contained in M and tangent to* $[v]$.

Proof. If F is analytically conjugated to an asymptotic ultra-resonant form $G = (G^x, G^y, G^z)$, then $G^z(x, y, 0) = 0$, i.e. $\{z = 0\}$ is invariant by G. For the converse, if there exists an invariant nonsingular complex manifold M of dimension $n - m$ for F passing through O and tangent to the eigenspace of 1, then we can choose local coordinates such that $M = \{z = 0\}$. In this system of coordinates F is of the form (F^x, F^y, F^z) such that $F^z(x, y, 0) = 0$ as M is invariant by F, which implies that F is asymptotically ultra-resonant.

Now assume that F is asymptotically ultra-resonant, and thus $M = \{z = 0\}$ is invariant by F. Then F restricted to M, denoted by F_M, is an analytic transformation of $\mathbf{C}^{n-m}$ tangent to the identity. By the definition of $\nu(F)$ and Lemma 2.3, we have $\nu(F_M) = \nu(F)$, where $\nu(F_M)$ is the order of F_M defined in Definition 1.2. Thus if $\nu(F) = \infty$, then $F_M(w) = w$, i.e. M is pointwise fixed by F. If $\nu(F) < \infty$, then it is easy to see that $[v] = [1 : v_0 : 0]$ is a non-degenerate characteristic direction for F if and only if $[1 : v_0]$ is a non-degenerate characteristic direction for F_M in the sense of Definition 1.2. Therefore we can conclude using Theorem 1.3. □

Remark 2.6. In this paper, we are only interested in the case $\nu(F) < \infty$. When $\nu(F) = \infty$, the dynamics of F may be very different (see [13]).

Before we go further, we need the following lemma.

Lemma 2.7. *Let F be as in* (2.1)*. Write $\Lambda = \mathrm{diag}(\Lambda_1, \ldots, \Lambda_r)$, where $\Lambda_i = \lambda_{(i)} I_{m_i}$ with $\lambda_{(i)}$'s mutually different and $m_1 + \cdots + m_r = m$. Similarly, write $z = (z_{(1)}, \ldots, z_{(r)})^t$ with $z_{(i)} = (z_{(i),1}, \ldots, z_{(i),m_i})$ and $z_1 = (z_{1,(1)}, \ldots, z_{1,(r)})^t$ with $z_{1,(i)} = (z_{1,(i),1}, \ldots, z_{1,(i),m_i})$. Then for any given integer $\mu > 1$ and any $1 \leqslant p \leqslant r$, there exists a local holomorphic change of coordinates such that in the expression of $z_{1,(p)}$ there are no terms $x^i y^j z^k$ with $2 \leqslant i + |j| + |k| \leqslant \mu$, $\lambda^k \neq \lambda_{(p)}$ and z^k not containing powers of $z_{(p),s}$ for any $1 \leqslant s \leqslant m_p$.*

Proof. For fixed $1 \leqslant p \leqslant r$ and $1 \leqslant s \leqslant m_p$, denote by $Z^{p,s}_{i,j,k}$ the non-zero coefficient of a term $x^i y^j z^k$ with the least $i + |j| + |k|$ such that $2 \leqslant i + |j| + |k| \leqslant \mu$, $\lambda^k \neq \lambda_{(p)}$ and z^k not containing powers of $z_{(p),s}$ for any $1 \leqslant s \leqslant m_p$ in the expression of $z_{1,(p),s}$. To get rid of such a term, consider the transformation

$$\begin{cases} x = X, \\ y = Y, \\ z_{(p')} = Z_{(p')}, \quad p' \neq p, \\ z_{(p),\tilde{s}} = Z_{(p),\tilde{s}}, \quad \tilde{s} \neq s, \\ z_{(p),s} = Z_{(p),s} + x^i y^j z^k S, \end{cases} \tag{2.8}$$

with $S = -(\lambda_{(p)} - \lambda^k)^{-1} Z^{p,s}_{i,j,k}$. Note that such a transformation only acts on that term and on terms of higher order. Proceeding this way, we are done. □

If $[v]$ is a non-degenerate characteristic direction, we can choose coordinates (x, y, z) in $\mathbf{C}^n = \mathbf{C} \times \mathbf{C}^l \times \mathbf{C}^m$, such that $[v] = [1 : v_0 : 0]$ and

$$F_{\nu(F)}\big([v]\big) = F_{\nu(F)}\big([1 : v_0 : 0]\big) = \big[p_{\nu(F)}\big([1 : v_0 : 0]\big) : q_{\nu(F)}\big([1 : v_0 : 0]\big) : 0\big]$$

with $p_{\nu(F)}([1 : v_0 : 0]) \neq 0$. By a further linear change of coordinates, we can assume that $[v] = [1 : 0 : 0]$. Performing changes of coordinates as in Lemmas 2.1 and 2.7, we can write F in the following form

$$\begin{cases} x_1 = x + p_{\nu(F)}(x, y, 0) + P(x, y, z) + O\big(\nu(F) + 1\big), \\ y_1 = y + q_{\nu(F)}(x, y, 0) + Q(x, y, z) + O\big(\nu(F) + 1\big), \\ z_1 = \Lambda z + R(x, y, z) + O\big(\nu(F) + 1\big), \end{cases} \tag{2.9}$$

with $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ analytic in $\mathbf{C}$, $\mathbf{C}^l$ and $\mathbf{C}^m$ respectively such that

(i) terms $x^i y^j z^k$ in $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ satisfy $2 \leqslant i + |j| + |k| \leqslant \nu(F)$ and $|k| \geqslant 1$;
(ii) writing $R(x, y, z) = (R_1(x, y, z), \ldots, R_r(x, y, z))$ as in Lemma 2.7, there are no terms $x^i y^j z_{(q),s}$ in $R_p(x, y, z)$ for any $1 \leqslant p \leqslant r$ and $q \neq p$.

Now we can make the following definition.

Definition 2.8. Let F be as in (2.9). We say that F is *dynamically separating* in the characteristic direction $[v] = [1 : 0 : 0]$ if $R_j(x, y, z)$ contains no terms $x^i z_{(j),s}$ with $i < \nu(F) - 1$ and $1 \leqslant s \leqslant m_j$ for any $1 \leqslant j \leqslant r$.

Remark 2.9. The above definition is needed (in our approach) because terms $x^i z_{(j),s}$ with $i < \nu(F) - 1$ in $R_j(x, y, z)$ are persistent under blow-ups. Though we do not know what happens in general if F is not dynamically separating in a characteristic direction, we will give a detailed study of such an example in Section 4.

The next lemma shows that the notion of dynamically separating is well defined (cf. [4, Lemma 2.5]).

Lemma 2.10. *Let F and G be two analytic transformations in ultra-resonant form and with the same characteristic direction* $[v]$. *If F is conjugated to G, then F is dynamically separating in the characteristic direction* $[v]$ *if and only if G has the same property.*

Proof. Without loss of generality, we can assume that both F and G are in the form (2.9) and $[v] = [1 : 0 : 0]$. Assuming that F is dynamically separating in the characteristic direction $[v]$, we need to show that G has the same property.

By Lemma 2.3, we have $\nu(F) = \nu(G) = \nu < \infty$. We can write F in the form

$$\begin{cases} x_1 = x + A_1(x, y) + B_1(x, y, z) + H_1(x, y, z), \\ y_1 = y + A_2(x, y) + B_2(x, y, z) + H_2(x, y, z), \\ z_1 = \Lambda z + A_3(x, z) + B_3(x, y, z) + H_3(x, y, z), \end{cases} \tag{2.10}$$

where $A_1(x, y)$ and $A_2(x, y)$ contain terms $x^i y^j$ with $i + |j| = \nu$ and $A_2(1, 0) = 0$, $A_3(x, z)$ contains terms $x^i z^k$ with $i = \nu - 1$ and $|k| = 1$, $B_h(x, y, z)$ and $H_h(x, y, z)$ contain other terms $x^i y^j z^k$ with $i + |j| + |k| \leqslant \nu$ and $i + |j| + |k| > \nu$ respectively, with $B_h(x, y, 0) = 0$, for $h = 1, 2, 3$.

Similarly, we can express G as

$$\begin{cases} \tilde{x}_1 = x + \tilde{A}_1(x, y) + \tilde{B}_1(x, y, z) + \tilde{H}_1(x, y, z), \\ \tilde{y}_1 = y + \tilde{A}_2(x, y) + \tilde{B}_2(x, y, z) + \tilde{H}_2(x, y, z), \\ \tilde{z}_1 = \Lambda z + \tilde{A}_3(x, z) + \tilde{B}_3(x, y, z) + \tilde{H}_3(x, y, z). \end{cases} \tag{2.11}$$

We need to show that $\tilde{A}_3(x, z)$ does not contain terms $x^i z^k$ with $i < \nu - 1$ and $|k| = 1$. More precisely, if we write $\tilde{A}_3(x, z) = (\tilde{A}_3^1, \ldots, \tilde{A}_3^r)$ as in Lemma 2.7, then we need to show that $\tilde{A}_3^p$ does not contain terms $x^i z_{(p),s}$ with $i < \nu - 1$ for any $1 \leqslant p \leqslant r$.

If Ψ is the transformation which conjugates F to G and preserves the direction $[v]$, then it is easy to see that Ψ must be of the following form

$$\begin{cases} x_1' = ax + \psi_1(x, y, z), \\ y_1' = Cy + \psi_2(x, y, z), \\ z_1' = Dz + \psi_3(x, y, z), \end{cases} \tag{2.12}$$

where $a \neq 0$, C is an invertible (l, l) matrix and $D = \operatorname{diag}(D_1, \ldots, D_r)$ with D_p being an invertible (m_p, m_p) matrix for $1 \leqslant p \leqslant r$ (as in Lemma 2.7).

From $F \circ \Psi = \Psi \circ G$, we obtain

$$\begin{aligned} &\Lambda \psi_3(x, y, z) + A_3(x_1', z_1') + B_3(x_1', y_1', z_1') + H_3(x_1', y_1', z_1') \\ &\quad = \psi_3(\tilde{x}_1, \tilde{y}_1, \tilde{z}_1) + D\big[\tilde{A}_3(x, z) + \tilde{B}_3(x, y, z) + \tilde{H}_3(x, y, z)\big]. \end{aligned} \tag{2.13}$$

Therefore,

$$\psi_3(\tilde{x}_1, \tilde{y}_1, \tilde{z}_1) - \Lambda \psi_3(x, y, z) = A_3(x_1', z_1') - D\tilde{A}_3(x, z) + R(x, y, z), \tag{2.14}$$

where $R(x, y, z) = (R_1(x, y, z), \ldots, R_r(x, y, z))$ does not contain terms $x^i z^k$ with $|k| = 1$ and $i < \nu$. Writing $\psi_3(x, y, z) = (\psi_3^1(x, y, z), \ldots, \psi_3^r(x, y, z))$, the above equation becomes

$$\psi_3^p(\tilde{x}_1, \tilde{y}_1, \tilde{z}_1) - \Lambda_p \psi_3^p(x, y, z) = A_3^p(x_1', z_1') - D_p \tilde{A}_3^p(x, z) + R_p(x, y, z), \quad 1 \leqslant p \leqslant r. \tag{2.15}$$

From the proof of Lemma 2.3, we know that $\psi_3(x, y, z)$ does not contain terms $x^i y^j$ with $i + |j| < \nu$. Writing $\psi_3(x, y, z) = \sum_{i+|j|+|k| \geqslant 2} \psi_3^{i,j,k} x^i y^j z^k$, one readily checks that

$$\psi_3^p(\tilde{x}_1, \tilde{y}_1, \tilde{z}_1) = \psi_3^p(x, y, \Lambda z) + R_p'(x, y, z), \tag{2.16}$$

where $R_p'(x, y, z)$ does not contain terms $x^i z_{(p),s}$ with $i < b + 1$, where $b = \min\{i\colon x^i z_{(p),s} \text{ in } \tilde{A}_3^p(x, z),\ 1 \leqslant p \leqslant r\}$.

Assume that $b < \nu - 1$. Then putting (2.16) into (2.15) we get

$$\psi_3^p(x, y, \Lambda z) - \Lambda_p \psi_3^p(x, y, z) = A_3^p(x_1', z_1') - D_p \tilde{A}_3^p(x, z) + \tilde{R}_p(x, y, z), \quad 1 \leqslant p \leqslant r, \tag{2.17}$$

where $\tilde{R}_p(x, y, z)$ does not contain terms $x^i z_{(p),s}$ with $i < b + 1$. But note that $\psi_3^p(x, y, \Lambda z) - \Lambda_p \psi_3^p(x, y, z)$ does not contain terms $x^i z_{(p),s}$ for any $1 \leqslant p \leqslant r$ and $1 \leqslant s \leqslant m_p$. Since $A_3^p(x_1', z_1')$ does not contain terms $x^i z_{(p),s}$ with $i < \nu - 1$, we get a contradiction for at least one of the r equations in (2.17). Therefore we must have $b \geqslant \nu - 1$. □

Now let F be as in (2.9) and assume that F is dynamically separating in the characteristic direction $[v] = [1 : 0 : 0]$. For simplicity, we will write ν for $\nu(F)$. Denote by $r_\nu(x, 0, z)$ the terms $x^i z^k$ with $i = \nu - 1$ and $|k| = 1$ in $R(x, y, z)$. Let us make the blow-up $y = xu$, with $u \in \mathbf{C}^l$ and $z = xv$, with $v \in \mathbf{C}^m$. Then F lifts to a map of the form

$$\begin{cases} x_1 = x + x^\nu p_\nu(1, u, 0) + \tilde{P}(x, u, v) + O(\nu + 1), \\ u_1 = u + x^{\nu-1} s(u) + \tilde{Q}(x, u, v) + O(\nu + 1) + O\big(x^\nu\big), \\ v_1 = \Lambda v + x^{\nu-1} t(u, v) + \tilde{R}(x, u, v) + O(\nu + 1) + O\big(x^\nu\big), \end{cases} \tag{2.18}$$

with $s(u) = q_\nu(1, u, 0) - p_\nu(1, u, 0)u$ and $t(u, v) = r_\nu(1, 0, v) - p_\nu(1, u, 0)\Lambda v$. Using the Taylor expansion and changing x into $[-p_\nu(1, 0, 0)]^{-1/(\nu-1)} x$, we can write the above equation as

$$\begin{cases} x_1 = x - x^\nu + \tilde{P}(x, u, v) + O\big(x^{\nu+1}, x^\nu \|w\|\big), \\ u_1 = \big[I_l - x^{\nu-1} \tilde{A}\big] u + \tilde{Q}(x, u, v) + O\big(x^{\nu-1} \|w\|^2, x^\nu \|w\|, x^\nu\big), \\ v_1 = \big[\Lambda - x^{\nu-1} \tilde{B}\big] v + \tilde{R}(x, u, v) + O\big(x^{\nu-1} \|w\|^2, x^\nu \|w\|, x^\nu\big), \end{cases} \tag{2.19}$$

where $w=(u,v)$, $\tilde{A}=s'(0)/p_\nu(1,0,0)$ and $\tilde{B}=t_v(0,0)/p_\nu(1,0,0)$. Note that $s(0)=0$, $t(0,0)=0$ and $t_u(0,0)=0$. Here $\tilde{P}$, $\tilde{Q}$ and $\tilde{R}$ have the same properties as P, Q and R in (2.9).

Remark 2.11. By Lemma 2.7 and the definition of $\tilde{B}$, it is easy to see that $\tilde{B}=\operatorname{diag}(\tilde{B}_1,\dots,\tilde{B}_r)$ with $\tilde{B}_p$ being an invertible (m_p,m_p) matrix for $1\leqslant p\leqslant r$ (as in Lemma 2.7).

Remark 2.12. By Definition 2.4, a transformation which fixes the origin and preserves the characteristic direction $[1:0:0]$ sends $\{z=0\}$ and $\{y=0\}$ into themselves. By the definition of $\tilde{A}$ and $\tilde{B}$, one readily checks that they are only affected by the linear part of such a transformation. We have then to inspect the effect of transformations of the type

$$\begin{cases} x=X+O(\|Y,Z\|),\\ y=LY,\\ z=MZ, \end{cases} \tag{2.20}$$

where L is an invertible (l,l) matrix and M is an invertible (m,m) matrix. It is easy to check that such a transformation will transform Eq. (2.19) to the form

$$\begin{cases} X_1=X-X^\nu+\tilde{P}(X,U,V)+O(X^{\nu+1},X^\nu\|W\|),\\ U_1=L^{-1}[I_l-X^{\nu-1}\tilde{A}]LU+\tilde{Q}(X,U,V)+O(X^{\nu-1}\|W\|^2,X^\nu\|W\|,X^\nu),\\ V_1=M^{-1}[\Lambda-X^{\nu-1}\tilde{B}]MV+\tilde{R}(X,U,V)+O(X^{\nu-1}\|W\|^2,X^\nu\|W\|,X^\nu), \end{cases} \tag{2.21}$$

with $U=Y/X$, $V=Z/X$ and $W=(U,V)$. Therefore, one can assume that in Eq. (2.19) both $\tilde{A}$ and $\Lambda^{-1}\tilde{B}$ are in Jordan canonical forms.

The following simple lemma is similar to Lemma 2.1.

Lemma 2.13. *Let F be as in* (2.19). *Then for any given integer $\mu>1$ and any $1\leqslant p\leqslant l$, there exists a local holomorphic change of coordinates such that in the expression of $u_{1,p}$ there are no terms $x^iu^jv^k$ with $2\leqslant i+|j|+|k|\leqslant\mu$, $\lambda^k\neq 1$ and u^j not containing powers of u_p.*

Proof. For a fixed $1\leqslant p\leqslant l$, denote by $U^p_{i,j,k}$ the non-zero coefficient of a term $x^iu^jv^k$ with the least $i+|j|+|k|$ such that $2\leqslant i+|j|+|k|\leqslant\mu$, $\lambda^k\neq 1$ and u^j not containing powers of u_p in the expression of $u_{1,p}$. To get rid of such a term, consider the transformation

$$\begin{cases} x=X,\\ u_{p'}=U_{p'},\quad p'\neq p,\\ u_p=U_p+x^iu^jv^kS,\\ v=V, \end{cases} \tag{2.22}$$

with $S=-(1-\lambda^k)^{-1}U^p_{i,j,k}$. Note that such a transformation only acts on that term and on terms of higher order. Proceeding this way, we are done. $\square$

We can now make the last simplification in the next proposition.

Proposition 2.14. *Let F be as in* (2.19). *For any $\mu\geqslant\nu+2$, one can perform a finite number of blow-ups and changes of coordinates such that the resulting map is given by*

$$\begin{cases} x_1=x-x^\nu+O(x^\nu\|w\|,x^{\nu+1}\log x),\\ u_1=(I_l-x^{\nu-1}A)u+O(x^{\nu-1}\|w\|^2,x^\nu\log x\|w\|,x^\mu(\log x)^{q_\mu}),\\ v_1=(\Lambda-x^{\nu-1}B)v+O(x^{\nu-1}\|w\|^2,x^\nu\log x\|w\|,x^\mu(\log x)^{q_\mu}), \end{cases} \tag{2.23}$$

where q_μ is a fixed positive integer and both A and $\Lambda^{-1}B$ are in Jordan canonical forms.

Proof. First, we need to get rid of all the terms in $\tilde{P}$, $\tilde{Q}$ and $\tilde{R}$. To do so, we apply Lemmas 2.7 and 2.13 first. Then, if necessary, we make enough number of blow-ups. Note that the order $i+|j|+|k|$ of a term $x^i u^j v^k$ in $\tilde{P}$, $\tilde{Q}$ and $\tilde{R}$ will increase at least by one after each blow-up, since F is dynamically separating in the characteristic direction $[1:0:0]$.

Second, we need to push pure x terms to higher order. For this, we apply Lemma 2.1 and [7, Proposition 3.5].

Let d be the least number of blow-ups needed, which is determined by the least of $|j|+|k|$ of terms $x^i u^j v^k$ in $\tilde{P}$ and $|j|+|k|-1$ of terms $x^i u^j v^k$ in $\tilde{Q}$ and $\tilde{R}$ (after applying Lemmas 2.7 and 2.13). Then in the new system of coordinates, the transformation F takes the form (2.23), with $A=\tilde{A}-dI_l$ and $B=\tilde{B}-d\Lambda$. By Remark 2.12, we can assume without loss of generality that both A and $\Lambda^{-1}B$ are in Jordan canonical forms. □

Remark 2.15. When applying [7, Proposition 3.5], there may be functions of $\log x$ involved. While this seems formal at this point, it will not cause any problem in the future as we will work in a region where $\log x$ is well defined.

Remark 2.16. From Remark 2.12, we know that the classes of similarity of $\tilde{A}$ and $\tilde{B}$ are invariant under changes of coordinates. It is also easy to see that the number of blow-ups d in the above proposition is only affected by the linear part of a transformation and that a transformation as in (2.20) does not change d. Hence, the proof of the above proposition shows that the eigenvalues of A and $\Lambda^{-1}B$ are invariants associated to the characteristic direction $[1:0:0]$ of F.

3. Parabolic curves

We prove Theorem 1.5 in this section. By the discussion in the previous section, we will work with analytic transformations of the form (2.23).

Set $N=\operatorname{diag}(I_l,\Lambda)$, $M=\operatorname{diag}(A,B)$ and $L=N^{-1}M$. Let $\{\beta_i\}_{1\leqslant i\leqslant n-1}$ be the eigenvalues of L and let $\beta=\max_i\{\operatorname{Re}\beta_i\}$. Choose μ in (2.23) such that $\mu>\nu+\beta$. Let us rewrite (2.23) in the following form

$$\begin{cases} x_1=f(x,w)=x-x^\nu+O\big(x^\nu\|w\|,x^{\nu+1}\log x\big),\\ w_1=\Psi(x,w)=\big(N-x^{\nu-1}M\big)w+O\big(x^{\nu-1}\|w\|^2,x^\nu\log x\|w\|,x^\mu(\log x)^{q_\mu}\big).\end{cases}\tag{3.1}$$

Let $D_\rho=\{x\in\mathbf{C}\colon |x^{\nu-1}-\rho|<\rho\}$, with $\rho>0$ small. Let B_ρ be the Banach space of functions $\varphi(\cdot)=x^2h(\cdot)$ with h holomorphic bounded from D_ρ to $\mathbf{C}^{n-1}$, endowed with the norm $\|\varphi\|_\rho=\|h\|_\infty$. For $\varphi\in B_\rho$, let $f_\varphi(x)=f(x,\varphi(x))$. The classical results of Fatou imply that, for ρ small enough, f_φ maps each component of D_ρ into itself and that we have $|x_n|=|f_\varphi^n(x)|=O(1/n^{1/(\nu-1)})$. If we find $\varphi\in B_\rho$ such that

$$\varphi\big(f\big(x,\varphi(x)\big)\big)=\Psi\big(x,\varphi(x)\big),\tag{3.2}$$

with $\lim_{x\to0}\varphi(x)=0$, then $(x,\varphi(x))$ restricted to each connected component of D_ρ will yield a parabolic curve for F tangent to $[v]=[1:0:0]$ (via blowing-down).

Write $x^L=\exp(L\log x)$, which is well defined on each connected component of D_ρ. By (3.1), we have

$$x_1^{-L}=x^{-L}\big(I+x^{\nu-1}L+O\big(x^{\nu-1}\|w\|,x^\nu\log x\big)\big).$$

Therefore,

$$\begin{aligned} x_1^{-L}N^{-1}w_1&=x^{-L}\big(I+x^{\nu-1}L+O\big(x^{\nu-1}\|w\|,x^\nu\log x\big)\big)N^{-1}\\ &\quad\times\big(\big(N-x^{\nu-1}M\big)w+O\big(x^{\nu-1}\|w\|^2,x^\nu\log x\|w\|,x^\mu(\log x)^{q_\mu}\big)\big)\\ &=x^{-L}w+O\big(x^{\nu-1}\|w\|^2,x^\nu\log x\|w\|,x^\mu(\log x)^{q_\mu}\big).\end{aligned}\tag{3.3}$$

We need the following lemma.

Lemma 3.1. *The equality $x_i^{-L}N^{-1}=N^{-1}x_i^{-L}$ holds for all $i\geqslant1$.*

Proof. Using the notion of Lemma 2.7, we can write $N=\operatorname{diag}(I_l,\Lambda_1,\dots,\Lambda_r)$. By Remark 2.11 and Proposition 2.14, we can write $M=\operatorname{diag}(A,B_1,\dots,B_r)$, with B_p being an invertible (m_p,m_p) matrix for $1\leqslant p\leqslant r$

(as in Lemma 2.7). Therefore $L = \operatorname{diag}(A, L_1, \ldots, L_r)$ with $L_p = \Lambda_p^{-1} B_p$, $1 \leqslant p \leqslant r$. Hence $x_i^{-L} = \operatorname{diag}(x_i^{-A}, x_i^{-L_1}, \ldots, x_i^{-L_r})$ for all $i \geqslant 1$. Since $\Lambda_p = \lambda_{(p)} I_{m_p}$ for $1 \leqslant p \leqslant r$, the equality follows. □

By the above lemma, we can write Eq. (3.3) as

$$N^{-1} x_1^{-L} w_1 = x^{-L} w + O\big(x^{\nu-1}\|w\|^2, x^{\nu}\log x\|w\|, x^{\mu}(\log x)^{q_\mu}\big).$$

Define

$$H(x, w) := x^L\big(x^{-L}w - N^{-1}x_1^{-L}w_1\big) = w - x^L N^{-1} x_1^{-L} w_1. \tag{3.4}$$

Then

$$H(x, w) = O\big(x^{\nu-1}\|w\|^2, x^{\nu}\log x\|w\|, x^{\mu}(\log x)^{q_\mu}\big),$$

and the functional equation (3.2) is equivalent to

$$x^{-L}\varphi(x) - N^{-1}x_1^{-L}\varphi(x_1) = x^{-L}H\big(x, \varphi(x)\big). \tag{3.5}$$

Let T be the operator on B_ρ defined by

$$T\varphi(x) = x^L \sum_{i=0}^{\infty} N^{-i} x_i^{-L} H\big(x_i, \varphi(x_i)\big). \tag{3.6}$$

One can show exactly as in [7] that the series converges normally, if φ is chosen so that $x^{\nu-1}\|\varphi\|^2$ and $x^{\nu}\log x\|\varphi\|$ are $o(x^{\mu}(\log x)^{q_\mu})$, and that $T\varphi \in B_\rho$. Moreover, the argument of Hakim carries over to our case, as the spectrum of dF_O lies in the unit circle, and shows that T restricted to a certain closed subset of B_ρ is continuous and contracting. Hence T has a fixed point. It is clear that such a fixed point is a solution to (3.5), thus to (3.2).

4. A non-dynamically-separating example

As suggested in [4], we study the analytic transformation of $\mathbf{C}^2$ of the form

$$\begin{cases} z_1 = z - z^3, \\ w_1 = \lambda w - azw + z^{d+1}, \end{cases} \tag{4.1}$$

where $\lambda = e^{i\theta}$, $a = |a|e^{i\phi} \neq 0$ and $d \geqslant 3$. Note that $[1:0]$ is a degenerate characteristic direction for this map. If $\lambda \neq 1$, then the map is quasi-parabolic and *non-dynamically-separating* along $[1:0]$.

Remark 4.1. The reason that we need $d \geqslant 3$ will be clear from what follows. On the other hand, we can always make changes of coordinates to get rid of terms z^{d+1} with $d = 1, 2$ in the expression of w_1. To get rid of the term bz^{d+1}, we make the change of coordinates $(z = \tilde{z},\ w = \tilde{w} + (b/a)\tilde{z}^d)$ if $\lambda = 1$, and $(z = \tilde{z},\ w = \tilde{w} + (b/(1-\lambda))\tilde{z}^d)$ if $\lambda \neq 1$.

We blow-up the origin O. In the new coordinates $(z = z,\ w = zu)$, the map takes the form

$$\begin{cases} z_1 = z - z^3, \\ u_1 = \dfrac{\lambda - az}{1 - z^2} u + \dfrac{z^d}{1 - z^2}. \end{cases} \tag{4.2}$$

It is easy to check that

$$u_n = \prod_{j=0}^{n-1} \frac{\lambda - az_j}{1 - z_j^2} u + \sum_{i=0}^{n-1} \prod_{j=i+1}^{n-1} \frac{\lambda - az_j}{1 - z_j^2} \frac{z_i^d}{1 - z_i^2}. \tag{4.3}$$

By the classical theory of Fatou, we have $z_n^2 = O(1/n)$ for n large. Therefore $z_n^2 = b_n \cdot 1/n \cdot e^{i\psi_n}$ where $b_n > 0$ goes to 1 and ψ_n goes to 0 as n goes to infinity. For $\delta > 0$ small, let $D_1 = \{z\colon |z^2 - \delta| < \delta,\ \operatorname{Re} z > 0\}$ and $D_2 = \{z\colon |z^2 - \delta| < \delta,\ \operatorname{Re} z < 0\}$. Write $c_j = \frac{\lambda - az_j}{1 - z_j^2}$, then for $z \in D_1$ we have $z_j \in D_1$ for all $j \geqslant 1$ and

$$
\begin{aligned}
|c_j| &= \frac{|\lambda - az_j|}{|1 - z_j^2|} = \frac{|e^{i\theta} - |a|e^{i\phi}(\sqrt{b_j}/\sqrt{j}\,)e^{i\psi_j/2}|}{|1 - (b_j/j)e^{i\psi_j}|} \\
&= \frac{|e^{i(\theta-\phi-\psi_j/2)} - |a|\sqrt{b_j}/\sqrt{j}|}{|e^{-i\psi_j} - b_j/j|} \\
&= \frac{|\cos(\theta-\phi-\psi_j/2) - |a|\sqrt{b_j}/\sqrt{j} + i\sin(\theta-\phi-\psi_j/2)|}{|\cos(\psi_j) - b_j/j - i\sin(\psi_j)|} \\
&= \frac{[1 - 2|a|(\sqrt{b_j}/\sqrt{j}\,)\cos(\theta-\phi-\psi_j/2) + |a|^2 b_j/j]^{1/2}}{[1 - 2(b_j/j)\cos(\psi_j) + b_j^2/j^2]^{1/2}},
\end{aligned}
\tag{4.4}
$$

and similarly for $z \in D_2$ we have $z_j \in D_2$ for all $j \geqslant 1$ and

$$
|c_j| = \frac{[1 + 2|a|(\sqrt{b_j}/\sqrt{j}\,)\cos(\theta-\phi-\psi_j/2) + |a|^2 b_j/j]^{1/2}}{[1 - 2(b_j/j)\cos(\psi_j) + b_j^2/j^2]^{1/2}}. \tag{4.5}
$$

There are two different cases.

In the first case, we have $\mathrm{Re}(\lambda \cdot \bar{a}) > 0$, i.e. $\cos(\theta - \phi) > 0$. Then for small $\epsilon > 0$ we can choose j large enough such that $\cos(\theta - \phi - \psi_j/2) > \epsilon$. For $z \in D_1$, we get from (4.4) that

$$
\begin{aligned}
|c_j| &< \frac{(1 - 2|a|\epsilon\sqrt{b_j}/\sqrt{j} + |a|^2 b_j/j)^{1/2}}{(1 - 2b_j/j)^{1/2}} \\
&= \big(1 - 2|a|\epsilon\sqrt{b_j}/\sqrt{j} + |a|^2 b_j/j\big)^{1/2}(1 + 2b_j/j + \cdots)^{1/2} \\
&< \big(1 - |a|\epsilon/2 \cdot 1/j\big)^{1/2}
\end{aligned}
$$

for j large. Since $1 - |a|\epsilon/2 \cdot 1/j = O((1 - 1/j)^{|a|\epsilon/2})$ and $\prod_{j=i+1}^{n}(1 - 1/j) = i/n$, we have $\prod_{j=i+1}^{n} c_j = O((i/n)^{|a|\epsilon/4})$. Set $\alpha = d/2 - |a|\epsilon/4$. We can assume that $\alpha > 1$ since $d \geqslant 3$ and ϵ is small. Since

$$
\sum_{i=1}^{n} \frac{1}{i^\alpha} < 1 + \int_1^n x^{-\alpha}\,dx = \frac{\alpha}{\alpha - 1} - \frac{1}{\alpha - 1} n^{1-\alpha} < \frac{\alpha}{\alpha - 1},
$$

we get from (4.3) that $u_{n+1} = O(1/n^{|a|\epsilon/4})u + O(1/n^{|a|\epsilon/4})$. Hence u_n goes to 0 as n goes to infinity. Therefore there is a two-dimensional parabolic domain over D_1 tangent to the direction $[1:0]$.

For $z \in D_2$, we get from (4.5) that

$$
|c_j| > \big(1 + 2|a|\epsilon\sqrt{b_j}/\sqrt{j}\big)^{1/2} > \big(1 + |a|\epsilon \cdot 1/j\big)^{1/2}
$$

for j large. Since $1 + |a|\epsilon \cdot 1/j = O((1 + 1/j)^{|a|\epsilon})$ and $\prod_{j=n}^{i}(1 + 1/j) = (i+1)/n$, we have $(\prod_{j=n}^{i} c_j)^{-1} = O((n/i)^{|a|\epsilon/2})$. Choose

$$
u = -\sum_{i=0}^{\infty}\left(\prod_{j=0}^{i}\frac{\lambda - az_j}{1 - z_j^2}\right)^{-1}\frac{z_i^d}{1 - z_i^2}, \tag{4.6}
$$

then from (4.3) we get

$$
u_n = -\sum_{i=n}^{\infty}\left(\prod_{j=n}^{i} c_j\right)^{-1}\frac{z_i^d}{1 - z_i^2}.
$$

Since

$$
\sum_{i=n+1}^{\infty}\frac{1}{i^{d/2+|a|\epsilon/2}} < \int_n^{\infty} x^{-d/2-|a|\epsilon/2}\,dx = \frac{(d/2 + |a|\epsilon/2 - 1)^{-1}}{n^{d/2+|a|\epsilon/2-1}},
$$

we have $u_{n+1} = O(1/n^{d/2-1})$. Hence u_n goes to 0 as n goes to infinity. One readily checks that the series in (4.6) converges and that $u(z)$ is invariant, i.e. $u(z_1) = u_1(z)$. Therefore (4.6) gives a parabolic curve over D_2 tangent to the direction $[1:0]$.

In the second case, we have $\mathrm{Re}(\lambda \cdot \bar{a}) < 0$. Then arguing as above, we conclude that there is a parabolic curve defined by (4.6) over D_1 and a two-dimensional parabolic domain over D_2, both tangent to the direction $[1:0]$.

Remark 4.2. As already noted, when $\lambda \neq 1$, the map (4.1) is quasi-parabolic and non-dynamically-separating along $[1:0]$. The above discussion shows that it is still possible for the existence of parabolic curves in this case. On the other hand, if $\lambda = 1$, then the map (4.1) is tangent to the identity with $[1:0]$ being a degenerate characteristic direction. In [2], Abate introduced an invariant, called *residual index*, and showed the existence of parabolic curves when the index associated to a characteristic direction does not belong to $\mathbf{Q}^+$. Later in [9], Molino generalized this result to the case when the index associated to a characteristic direction is non-zero, using ideas from both [2] and [7]. However, the index associated to the characteristic direction $[1:0]$ for our example is zero. Therefore, the parabolic curves we found above are "new" even in the tangent to the identity case.

Remark 4.3. While in the non-degenerate case for a map tangent to the identity or the dynamically separating case for a quasi-parabolic map, the parabolic manifolds are of the same dimension, the above example shows that this might not be true in the degenerate case for a map tangent to the identity or the non-dynamically-separating case for a quasi-parabolic map. We suspect that this phenomenon of *symmetry break-down* is universal in such cases.

Remark 4.4. In a forthcoming paper [14], we are going to extend the results of Hakim in [8] to the quasi-parabolic case.

Acknowledgment

The author would like to thank the referee for many useful comments and suggestions.

References

[1] M. Abate, Discrete local holomorphic dynamics, in: Proc. of 13th Seminar on Anal. and Its Appl., Isfahan, March 2003.
[2] M. Abate, The residual index and the dynamics of holomorphic maps tangent to the identity, Duke Math. J. 107 (2001) 173–207.
[3] F. Bracci, Local dynamics of holomorphic diffeomorphisms, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 7 (2004) 609–636.
[4] F. Bracci, L. Molino, The dynamics near quasi-parabolic fixed points of holomorphic diffeomorphisms in $\mathbf{C}^2$, Amer. J. Math. 126 (2004) 671–686.
[5] P. Fatou, Substitutions analytiques et équations fonctionnelles à deux variables, Ann. Sci. École Norm. Sup. 40 (1924) 67–142.
[6] M. Hakim, Attracting domains for semi-attractive transformations of $\mathbf{C}^p$, Publ. Mat. 38 (1994) 479–499.
[7] M. Hakim, Analytic transformations of $(\mathbf{C}^p,0)$ tangent to the identity, Duke Math. J. 92 (1998) 403–428.
[8] M. Hakim, Transformations tangent to the identity. Stable pieces of manifolds, preprint, 1998.
[9] L. Molino, The dynamics of maps tangent to the identity and with non-vanishing index, Trans. Amer. Math. Soc., in press.
[10] Y. Nishimura, Automorphismes analytiques admettant des sous-variétés de points fixés attractives dans la direction transversale, J. Math. Kyoto Univ. 23 (1983) 289–299.
[11] J. Pöschel, On invariant manifolds of complex analytic mappings near fixed points, Expo. Math. 4 (1986) 97–109.
[12] M. Rivi, Parabolic manifolds for semi-attractive holomorphic germs, Michigan Math. J. 49 (2001) 211–241.
[13] F. Rong, Linearization of holomorphic germs with quasi-parabolic fixed points Ergodic Theory Dynam. Systems, in press.
[14] F. Rong, Quasi-parabolic analytic transformations of $\mathbf{C}^n$. Parabolic manifolds, preprint, 2007.
[15] T. Ueda, Local structure of analytic transformations of two complex variables, I, J. Math. Kyoto Univ. 26 (1986) 233–261.
[16] T. Ueda, Local structure of analytic transformations of two complex variables, II, J. Math. Kyoto Univ. 31 (1991) 695–711.
[17] B.J. Weickert, Attracting basins for automorphisms of $\mathbf{C}^2$, Invent. Math. 132 (1998) 581–605.